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ORIENTED MANIFOLDS WITH SMOOTH FAMILIES AND MODULE OF SKEW LINEAR MAPS

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Abstract

Some basic properties of smooth families of differential forms and oriented n-manifolds are developed in this paper. The module of skew p-linear maps $A^p(M)$ from $\mathcal{X}(M)$to $\mathcal{S}(M)$ is extended to $A^0(M)$ by putting $i(X)f = 0, f \in \mathcal{S}(M)$. The set of smooth families of p-forms $\{A_t^p(M)\}_{t \in \mathbb{R}}$ on M is the set of smooth families of cross-sections in the vector bundle $\wedge^p T_M^*$ and $\{A_t^p(M)\}_{t \in \mathbb{R}} = A^{0,p}(\mathbb{R} \times M)$. Every smooth family of p-forms on M is homogeneous of bidegree $(0, p)$ and has a differential form on $\mathbb{R} \times M$. For a gradientδf and one-form ω, we have $i(X)\delta f = X(f)$ and $i(X)\omega = \langle \omega, X \rangle$ respectively. Finally, a graded δ-stable ideal $A_M(\mathbb{R} \times M) \subset A(\mathbb{R} \times M)$ is defined for an oriented n-manifold M and it is shown that $\int_{I \times M} \delta\Phi = \int_M j_b^* \Phi - \int_M j_a^* \Phi$ for $\Phi \in A_M^n(\mathbb{R} \times M)$ and $\int_B \delta\Phi = \int_S i^* \Phi$ for $\Phi \in A^n(U)$.

***Keywords*:** Manifolds, vector bundle, smooth function, differential form, skew p-linear maps.

Introduction

Let $\{(U_\alpha, u_\alpha) : \alpha \in I\}$ be an atlas for a topological manifold M. Let U_α, U_β be two neighbourhoods such that $U_{\alpha\beta} = U_\alpha \cup U_\beta \neq \emptyset$. Then, a homeomorphism $u_{\alpha\beta} : u_\alpha(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ is defined by $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}$. This map is known as the identification map for U_α and U_β (Bishop and Crittenden, 1964). Also, $u_{\gamma\beta} \circ u_{\beta\alpha} = u_{\gamma\alpha}$ in $u_\alpha(U_{\alpha\beta\gamma})$ and $u_{\alpha\alpha}(x) = x$, $x \in u_\alpha(U_\alpha)$. If all the identification maps of an atlas $\{(U_\alpha, u_\alpha)\}$ are smooth, then the atlas $\{(U_\alpha, u_\alpha)\}$is called smooth (Hoffman and Spruck, 1974). Two smooth atlases $\{(U_\alpha, u_\alpha)\}$ and $\{(V_i, v_i)\}$ are said to be equivalent if all the maps

$$v_i \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap V_i) \to v_i(U_\alpha \cap V_i)$$

and their inverses are smooth. Every smooth structure on M is an

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equivalence class of smooth atlases on M and a topological manifold with a smooth structure is called a smooth manifold (Olum, 1953).

Consider two manifolds M, N and let $\varphi: M \to N$ be a continuous map. Assume that $\{(U_\alpha, u_\alpha)\}$ and $\{(V_i, v_i)\}$ are atlases for M and N respectively. Then φ defines continuous maps $\varphi_{i\alpha} : u_\alpha(U_\alpha \cap \varphi^{-1}(V_i)) \to v_i(V_i)$ by

$$\varphi_{i\alpha} = v_i \circ \varphi \circ u_\alpha^{-1}.$$

If the maps $\varphi_{i\alpha}$ are smooth, then $\varphi: M \to N$ is said to be smooth. This definition does not depend on the choice of atlases for M and N. Also, $\mu \circ \varphi: M \to P$ is smooth if the maps $\varphi: M \to N$ and $\mu: N \to P$ are smooth (Narasimhan, 1968). The set of smooth maps from M to N is denoted by $\mathcal{S}(M; N)$. If f and g are two smooth functions on a manifold M, then smooth functions $\lambda f + \mu g$ and fg are defined as follows

$$(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x), \qquad \lambda, \mu \in \mathbb{R}$$

$$(fg)(x) = f(x)\, g(x),\ x \in M.$$

These operations relate the set of smooth functions on M to an algebra over $\mathbb{R}$ and this is denoted by $\mathcal{S}(M)$. Assume that $\{U_\alpha\}$ is a locally finite family of open sets of M, and let $f_\alpha \in \mathcal{S}(M)$ satisfy the condition $carr\, f_\alpha \subset U_\alpha$. Then, there is a neighbourhood $V(a)$ which meets only finitely many of the U_α for each $a \in M$(Block and Weinberger, 1999). Consequently, $\sum_\alpha f_\alpha$ is a finite sum in this neighbourhood and a smooth function f on M is defined as follows

$$f(x) = \sum_\alpha f_\alpha(x)\,, x \in M.$$

Let T_M be a tangent bundle, then a vector field X on a manifold M is a cross-section (Kobayashi and Nomizu, 1963) in T_M. Therefore, a tangent vector $X(x)$ is assigned to every point $x \in M$ by a vector field X such that the map $M \to T_M$ is smooth. A module over the ring $\mathcal{S}(M)$ is formed by the vector fields on M and is denoted by $\mathcal{X}(M)$. Let ξ be a vector bundle. A cross-section σ in ξ is a smooth map $\sigma: B \to E$ satisfying $\pi \circ \sigma = \iota$. For every vector bundle ξ, there is a zero cross-section o defined by $o(x) = 0_x \in F_x, x \in B$.

The substitution operator, the Lie and exterior derivatives

Assume that $\Phi\ (p \geq 1)$ is a p-form and X is a vector field on a manifold M. A $(p - 1)$-form $i(X)\Phi$ is defined by

$$(i(X)\Phi)\big(X, X_1, \cdots, X_{p-1}\big) = \Phi\big(X, X_1, \cdots, X_{p-1}\big), \text{where} X_i \in \mathcal{X}(M),$$

or, equivalently,

$$(i(X)\Phi)\big(x;\ \xi_1,\ \cdots\ ,\xi_{p-1}\big) = \ \Phi\big(x; X(x), \xi_1,\ \cdots\ ,\xi_{p-1}\big),$$

where $x \in M,\ \xi_i \in T_x(M)$.

We consider $A^p(M)$as the module of skew p-linear maps from $\mathcal{X}(M)$ to $\mathcal{S}(M)$. The definition can be extended to $A^0(M)$ by putting$i\,(X)f = 0,\quad f \in \mathcal{S}(M)$ (Gromov and Lawson, 1980). If ω is a one-form, then we get $i(X)\omega = \langle \omega, X\rangle$. In particular, for a gradient δf, we have $i(X)\delta f = X(f)$. The map $i(X)\colon A(M) \to A(M)$ defined in the above way is called the substitution operator induced by X. This operator is homogeneous of degree -1, and satisfies the following equations

$$i(X)(f\ \cdot\ \Phi\ + g\ \cdot\ \Psi) = f\ \cdot i(X)\Phi + g\ \cdot i(X)\Psi$$

and

$$i(X)(\Phi\ \wedge\ \Psi) = \ i(X)(\Phi\ \wedge\ \Psi)$$

$$= \ i(X)\,\Phi\ \wedge\ \Psi + \ (-1)^p\ \Phi\ \wedge i(X)\Psi,$$

$$f, g \in \mathcal{S}(M),\quad \Phi\ \in\ A^p(M),\qquad \Psi\ \in A(M).$$

Consequently, $i(X)$is an antiderivation for each $X \in \mathcal{X}(M)$ in the algebra $A(M)$. If we consider a second vector field Y on M, we have

$$i(f\ \cdot\ X\ + g\ \cdot Y) = f\ \cdot i(X) + g\ \cdot i(Y) \text{ and}$$

$$i(X)i(Y) = \ -i(Y)i(X)\quad f, g\ \in \mathcal{S}(M).$$

Lemma 1. Let $\Phi \in A^p(M)$ $(p \geq 1)$ and $X \in \mathcal{X}(M)$. If Φ satisfies $i(X)\Phi = 0$ for every X, then $\Phi = 0$.

Consider a vector field $X \in \mathcal{X}(M)$ and a p-form $\Phi \in A^p(M)$ $(p \geq 1)$. We define a map $\mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{S}(M)$ by

$$\big(X_1, \cdots\ ,X_p\big) \mapsto X\left(\Phi\big(X_1, \cdots\ ,X_p\big)\right) - \sum_{j=1}^{p} \Phi\big(X_1, \cdots, [X,\qquad X_j],\ \cdots\ ,X_p\big).$$

This map is p-linear over $\mathbb{R}$ and skew-symmetric. Also, the relations

$$X(f\ \cdot g) = \ X(f)\cdot g + f\ \cdot X(g)$$

and

$$[X, f\cdot Y] = \ f\cdot[X,Y] + \ X(f)\cdot Y,\quad f, g \in \mathcal{S}(M)$$

indicate that it is p-linear over $\mathcal{S}(M)$. Therefore, it defines a p-form on M.

Definition 1. If $X \in \mathcal{X}(M)$, then the Lie derivative with respect to X is the real linear map $\theta(X): A(M) \to A(M)$ which is homogeneous of degree zero and given by

$$(\theta(X)\Phi)(X_1, \cdots, X_p) = X\left(\Phi(X_1, \cdots, X_p)\right) - \sum_{j=1}^{p} \Phi(X_1, \cdots, [X, X_j], \cdots, X_p),$$

where $\Phi \in A^p(M), p \geq 1, X_j \in \mathcal{X}(M)$, and $\theta(X)f = X(f), f \in \mathcal{S}(M)$.

Proposition 1. The Lie derivative has the following properties:

(1) $\theta(X)\,\delta f = \delta\theta(X) f = \delta X(f)$

(2) $\theta(X)(\Phi \wedge \Psi) = \theta(X)\Phi \wedge \Psi + \Phi \wedge \theta(X)\Psi$ $\qquad X, Y \in \mathcal{X}(M)$

(3) $\theta([X,Y]) = \theta(X)\theta(Y) - \theta(Y)\theta(X)$ $\qquad \Phi, \Psi \in A(M)$

(4) $\theta(f \cdot X) = f \cdot \theta(X) + \mu(\delta f)\, i(X)$

Here μ is the multiplication operator in $A(M)$ and $\mu(\Phi)\Psi = \Phi \wedge \Psi$.

Proof. From the definition of Lie derivative,

$$\langle \theta(X)\delta f, Y\rangle = X\big(Y(f)\big) - [X,Y](f)$$

$$= Y\big(X(f)\big) = \langle \delta(X(f)), Y\rangle, Y \in \mathcal{X}(M)$$

Thus, $\theta(X)\delta f = \delta\theta(X)f = \delta X(f)$ and (1) is proved.

Consider $\Phi \in A^p(M), \Psi \in A^p(M)$ and we will apply induction method on $p+q$. If $p+q=0$, then the derivation property of X on functions and (2) are equivalent. Now, let (2) be true for $p+q<k$. From definition, we have$i([X,Y]) = \theta(X)i(Y) - i(Y)\theta(X)$. So, for $p+q=k,\ X, Y \in \mathcal{X}(M)$, we can expand $i(Y)\theta(X)(\Phi \wedge \Psi)$ as follows

$$i(Y)\theta(X)(\Phi \wedge \Psi) = \theta(X)i(Y)(\Phi \wedge \Psi) - i([X,Y])(\Phi \wedge \Psi)$$

$$= \theta(X)[i(Y)\Phi \wedge \Psi + (-1)^p \Phi \wedge i(Y)\Psi - i([X,Y])\Phi \wedge \Psi - (-1)^p \Phi \wedge i([X,Y])\Psi$$

$$= \theta(X)i(Y)\Phi \wedge \Psi + i(Y)\Phi \wedge \theta(X)\Psi + (-1)^p \theta(X)\Phi \wedge i(Y)\Psi + (-1)^p \Phi \wedge \theta(X)i(Y)\Psi - i([X,Y])\Phi \wedge \Psi - (-1)^p \Phi \wedge i([X,Y])\Psi.$$

From the inductive hypothesis, we obtain the last equality. By the ant derivation rule for $i(Y)$ to this relation, we get

$$i(Y)\,\theta(X)(\,\Phi \wedge \Psi\,) = i(Y)[\theta(X)\Phi \wedge \Psi + \Phi \wedge \theta(X)\Psi],\;\; Y \in \mathcal{X}(M),$$

which implies that $\theta(X)(\Phi \wedge \Psi) = \theta(X)\Phi \wedge \Psi + \Phi \wedge \theta(X)\Psi$. Therefore, (2) is proved. Since $A(M)$ is generated as an algebra over $\mathbb{R}$ by functions and gradients, both sides of (3) are derivations in $A(M)$. It is sufficient to show that the effect of both sides of (3) on functions and gradients is the same. If we apply (3) to functions, then we obtain the definition of the Lie product. From (1) we get,

$$\begin{aligned}\theta([X,Y])\delta f &= \delta([X,Y]f)\\ &= \delta\Big(X\big(Y(f)\big) - Y\big(X(f)\big)\Big)\\ &= \theta(X)\theta(Y) - \theta(Y)\theta(X)\end{aligned}$$

Thus, $\theta([X,Y]) = \theta(X)\theta(Y) - \theta(Y)\theta(X)$for $\Phi, \Psi \in A(M)$, and (3) is proved.

We observe that both sides of (4) are derivations in $A(M)$. If we apply each side to $g \in \mathcal{S}(M)$ and δg, we obtain$f \cdot X(g)$ and $\delta(f \cdot X(g)) = f \cdot \delta(X(g)) + \delta f \wedge X(g)$ respectively. Hence (4) is proved.

Definition 2. Let $X \in \mathcal{X}(M)$. If $\theta(X) = 0$, then a differential form Φ is called invariant with respect to X. Since $\theta(X)$ is a derivation, the set of differential forms invariant with respect to X is a subalgebraof $A(M)$.

Assume that Φ is a p-form $(p \geq 1)$ on a manifold $M$and consider the map $\mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{S}(M)$ given by

$$\begin{aligned}(X_0, \cdots, X_p) \mapsto &\sum_{j=0}^{p} (-1)^j X_j \Big(\Phi\big(X_0, \cdots, \hat{X}_j, \cdots, X_p\big)\Big)\\ &+ \sum_{0 \leq i < j \leq p} (-1)^{i+j}\, \Phi\big([X_i, X_j], \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p\big).\end{aligned}$$

If $f, g \in \mathcal{S}(M)$and $X, Y \in \mathcal{X}(M)$, then the relations $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$ and $[X, f \cdot Y] = f \cdot [X, Y] + X(f) \cdot Y$ imply that this map is $(p+1)$-linear over $\mathcal{S}(M)$. It determines a $(p+1)$-form on M, because it is skew-symmetric (Hebey, 1996).

Definition 3. The exterior derivative is the $\mathbb{R}$-linear map $\delta: A(M) \to A(M)$ which is defined by

$$\delta\Phi(X_0, \cdots, X_p) = \sum_{j=0}^{p} (-1)^j X_j \left(\Phi(X_0, \cdots, \hat{X}_j, \cdots, X_p)\right)$$

$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \Phi([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p)$$

$$\Phi \in A^p(M), \; p \ge 1, \;\; X_j \in \mathcal{X}(M)$$

and

$$(\delta f)(X) = X(f), f \in \mathcal{S}(M), X \in \mathcal{X}(M).$$

The differential form $\delta\Phi$ is called the exterior derivative of Φ and it is homogeneous of degree 1 (Holopainen, 1992). If we combine the definition of the exterior derivative with that of the Lie derivative, we obtain a second expression for $\delta\Phi$ as follows:

$$\delta\Phi(X_0, \cdots, X_p) = \sum_{j=0}^{p} (-1)^j \left(\theta(X_j)\right) \Phi(X_0, \cdots, \hat{X}_j, \cdots, X_p)$$

$$- \sum_{i<j} (-1)^{i+j} \Phi([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p).$$

Proposition 2. The exterior derivative satisfies the following properties:

(1) $\theta(X) = i(X)\delta + \delta\, i(X), \quad X \in \mathcal{X}(M)$

(2) $\delta^2 = 0$

(3) $\delta\, \theta(X) = \theta(X)\delta.$

Proof. Let $X \in \mathcal{X}(M)$. From the definition of exterior derivative we easily get $\theta(X) = i(X)\delta + \delta i(X)$ which proves (1). δ^2 is a derivation, because δ is an antiderivation. Assume that $f \in \mathcal{S}(M)$. It is sufficient to show that $\delta^2 f = 0, \delta^2(\delta f) = 0$ because $A(M)$ is generated by functions and gradients and is an $\mathbb{R}$-algebra.

Consequently, we have

$$(\delta^2 f)(X, Y) = X(\langle \delta f, Y \rangle) - Y(\langle \delta f, X \rangle) - \langle \delta f, [X, Y] \rangle$$

$$= X\big(Y(f)\big) - Y\big(X(f)\big) - [X, Y]f = 0, \; X, Y \in \mathcal{X}(M).$$

That is, $\delta^2 f = 0$. So, it follows that $\delta^2 f(\delta f) = 0$. Hence (2) is proved.

If we use (2) and apply δ to both sides of (1), we easily get $\delta\, \theta(X) = \theta(X)\delta$. Hence (3) is proved.

Definition 4. If f is a smooth function on M, then the carrier (or support) of f is the closure of the set $\{x \in M: f(x) \neq 0\}$. This set is denoted by $carr\, f$. The carrier (or support) of a cross-section σ is the set defined by

$$carr\, \sigma = closure\, \{x \in B: \sigma(x) \neq 0_x\}.$$

Definition 5. Let Φ be a differential form on M, then Φ is said to have compact carrier, if $carr\, \Phi$ is compact. The set of differential forms on M with compact carrier is denoted by $A_c(M)$.

Definition 6. The partial exterior derivative with respect to M is the linear map $\delta_M: A(M \times N) \to A(M \times N)$ given by

$$(\delta_M \Omega)(Z_0, \cdots, Z_r) = \sum_{j=0}^{r} (-1)^j\, Z_j^M \left(\Omega\big(Z_0, \cdots, \hat{Z}_j, \cdots, Z_r\big)\right)$$
$$+ \sum_{0 \le i < j \le p} (-1)^{i+j}\, \Omega(\big(\mathrm{Z_i}, \mathrm{Z_j}\big)_{\mathrm{M}}, Z_0, \cdots, \hat{Z}_i, \cdots, \hat{Z}_j, \cdots, Z_r)$$

The partial exterior derivative with respect to N is given by

$$(\delta_N \Omega)(Z_0, \cdots, Z_r) = \sum_{j=0}^{r} (-1)^j\, Z_j^N \left(\Omega\big(Z_0, \cdots, \hat{Z}_j, \cdots, Z_r\big)\right)$$
$$+ \sum_{0 \le i < j \le p} (-1)^{i+j}\, \Omega(\big(\mathrm{Z_i}, \mathrm{Z_j}\big)_{N}, Z_0, \cdots, \hat{Z}_i, \cdots, \hat{Z}_j, \cdots, Z_r),$$
$$\Omega \in A^r(M \times N).$$

Both the partial exterior derivatives are homogeneous of degree 1.

Smooth family of cross-sections

Suppose that $\xi = (E, \pi, M, F)$ is a vector bundle and $\sigma: \mathbb{R} \to Sec\, \xi$ is a set map; that is, σ assigns a cross-section σ_i of ξ to every real number $t \in \mathbb{R}$. If a map $\mathbb{R} \times M \to E$ given by $\sigma(t, x) = \sigma_i(x)$ is smooth, then the map is called a smooth family of cross-sections (Yau, 1976). The set of smooth families of cross-sections in ξ is denoted by $\{Sec_t\, \xi\}_{t \in \mathbb{R}}$. For each fixed $x \in M$, each smooth family determines a smooth map $\sigma_x: \mathbb{R} \to F_x$ given by $\sigma_x(t) = \sigma(t, x)$.

Definition 7. For a smooth family of cross-sections σin ξ, the derivative of σisthe smooth family $\dot{\sigma}$ given by

$$\dot{\sigma}(t,x) = \lim_{s \to 0} \frac{\sigma(t+s,x) - \sigma(t,x)}{s}$$
$$= \frac{d}{ds}\sigma_x\Big|_{s=t}.$$

For $a \in \mathbb{R}$, the integral of σ is the smooth family $\int_a \sigma$ given by

$$\left(\int_a \sigma\right)(t,x) = \int_a^t \sigma_x(s)\,ds.$$

The definite integral $\int_a^b \sigma$ is the cross-section in ξ given by

$$\left(\int_a^b \sigma\right)(x) = \int_a^b \sigma_x(t)\,dt.$$

Using the fundamental theorem of calculus, we obtain the following relations for a smooth family σ:

$$\int_a^b \dot{\sigma}_t\, dt = \sigma_b - \sigma_a, \qquad a,b \in \mathbb{R}$$

and

$$\left(\int_a \sigma\right)^{\cdot}(t,x) = \sigma(t,x), \qquad t \in \mathbb{R},\ x \in M.$$

For the vector bundle $\wedge^p T_M^*$, a smooth family of p-forms on a manifold M is a smooth family of cross-sections in $\wedge^p T_M^*$ (Schoen and Yau, 1979). The set of smooth families of p-forms on M is denoted by $\left\{A_t^p(M)\right\}_{t \in \mathbb{R}}$ and $\left\{A_t^p(M)\right\}_{t \in \mathbb{R}} = A^{0,p}(\mathbb{R} \times M)$. Consequently, a smooth family of p-forms on M is homogeneous of bidegree $(0,p)$ and a differential form on $\mathbb{R} \times M$.

Proposition 3. Let M and N be two manifolds. If$\varphi: M \to N$ is a smooth map, and Φ is a smooth family of p-forms on N, then

$$((\iota \times \varphi)^* \Phi)^{\cdot} = (\iota \times \varphi)^* \dot{\Phi}$$

Proof. Let $t \in \mathbb{R}, y \in N$. Suppose that $\Phi_y: \mathbb{R} \to \wedge^p T_y(N)^*$ is the smooth map given by

$$\Phi_y(t) = \Phi(t,y).$$

Then, $\wedge^p (d\varphi)_x^* \circ \Phi_{\varphi(x)} : \mathbb{R} \to \wedge^p T_x(M)^*$ for $x \in M$ and

$$((\iota \times \varphi)^* \Phi)_x = \Lambda^p (d\varphi)_x^* \circ \Phi_{\varphi(x)}.$$

Since $\Lambda^p (d\varphi)_x^*$ is a linear map, it follows that

$$\frac{d}{ds}[(\iota \times \varphi)^* \Phi]_x = \Lambda^p (d\varphi)_x^* \circ \frac{d}{ds}\Phi_{\varphi(x)}.$$

That is,

$$[(\iota \times \varphi)^* \Phi]^{\cdot}_x = \Lambda^p (d\varphi)_x^* \circ \dot{\Phi}_{\varphi(x)} = [(\iota \times \varphi)^*]\dot{\Phi}_x .$$

Hence, $((\iota \times \varphi)^* \Phi)^{\cdot} = (\iota \times \varphi)^* \dot{\Phi}$, which completes the proof.

Theorem 1. If $X \in \mathcal{X}(M)$ and $\Phi$is a smooth family of p-forms on M, then

$$\delta \int_a^b \Phi_t \ dt = \int_a^b \delta\Phi_t \, dt.$$

Proof. If we use an atlas on M andreduce to the case M, then it will be a vector space E. As a result, we have

$A^{0,p}(\mathbb{R} \times E) = \mathcal{S}(\mathbb{R} \times E;\ \Lambda^p E^*).$

Let $a \in \Lambda^p E^*,\ f \in \mathcal{S}(\mathbb{R} \times E)$. Since both sides of $\delta \int_a^b \Phi_t \ dt = \int_a^b \delta\Phi_t \, dt$ are linear we may restrict to the case $\Phi(t,x) = f(t,x)a$. In the circumstance, $\delta \int_a^b \Phi_t \ dt = \int_a^b \delta\Phi_t \, dt$ is equivalent to

$$\sum_{\nu=1}^{n}\left[\frac{\partial}{\partial e^\nu}\int_a^b f(t,x) \ dt\right] e^{*\nu} \wedge a = \sum_{\nu}\left[\int_a^b \frac{\partial f}{\partial e^\nu}(t,x)dt\right] e^{*\nu} \wedge a,$$

where $e^{*\nu}, e^\nu$is a pair of dual bases for E^*, E and $\frac{\partial}{\partial e^\nu}$ denotes the partial derivative in the e^ν direction. Furthermore, it is evident that $\delta \int_a^b \Phi_t \, dt = \int_a^b \delta\Phi_t \, dt$. This completes the proof.

If $M$is a manifold and $\Omega \in A^p(\mathbb{R} \times M)$, then Ω can be uniquely decomposed in the form

$$\Omega = \Phi + \Psi, \ \text{where}\ \Phi \in A^{0,p}(\mathbb{R} \times M),\ \Psi \in A^{1,p-1}(\mathbb{R} \times M).$$

Assume that the smooth family of p-forms Φ, which satisfies $\Phi_t = j_t^*\Phi = j_t^*\Omega$. This smooth family will be denoted by $j^*\Omega$ and $(j^*\Omega)_t = j_t^*\Omega$.By integrating this family, we obtain the following differential form $I_a^b\Omega = \int_a^b (j_t^*\Omega) \, dt$ on M. The assignment $\Omega \mapsto I_a^b\Omega$ defines a linear map $I_a^b: A(\mathbb{R} \times M) \to A(M)$ which is homogeneous of degree zero (Wang and Zhang, 2011).

Lemma 2. Let Tdenotes the vector field $\frac{d}{dt}$ on $\mathbb{R}$ and $\Omega \in A(\mathbb{R} \times M)$; consider it as a vector field on $\mathbb{R} \times M$. Then

$$(j_t^* \Omega)_s^{\cdot} = j_s^* \, \theta(T)\Omega, \;\; \Omega \in A(\mathbb{R} \times M).$$

Proof. Let $a \in \wedge^p E^*$. Assume that $M$is a vector space $E$and that $\Omega \in A^{0,p}(\mathbb{R} \times E)$is of the form

$$\Omega(t,x) = f(t,x)a \text{ for } f \in \mathcal{S}(\mathbb{R} \times E).$$

Then

$$(j_t^* \Omega)_s^{\cdot}(x) = f'\left(s, x; \frac{d}{dt}\right) a$$

$$(\theta(T)\Omega)(s,x) = (j_t^* \theta(T)\Omega)(x).$$

Thus, $(j_t^* \Omega)_s^{\cdot} = j_s^* \, \theta(T)\Omega$, where $\Omega \in A(\mathbb{R} \times M)$.

Assume that X be a vector field on M which generates a one-parameter group of diffeomorphisms $\varphi: \mathbb{R} \times M \to M$. Then, $\varphi^* \Phi \in A(\mathbb{R} \times M)$ and $\varphi^* \theta(X)\Phi \in A(\mathbb{R} \times M)$for $\Phi \in A(M)$. If $\varphi_t : M \to M$ is the map $\varphi_t(x) = \varphi(t,x)$, then the corresponding smooth families of differential forms on $M$are given by

$$(j^* \varphi^* \Phi)_t = \varphi_t^* \, \Phi \text{ and } (j^* \varphi^* \theta(X)\Phi)_t = \varphi_t^* \theta(X)\Phi.$$

Proposition 4. If $\Phi \in A^p(M)$, then the family $\varphi_t^* \Phi$ satisfies the following relation:

$$\varphi_t^* \Phi - \Phi = \int_0^t (\varphi_s^* \theta(X)\Phi) \, ds.$$

In particular, $(\varphi_t^* \Phi)_0^{\cdot} = \theta(X)\Phi$.

Proof. It is evident that $T \sim X$. It implies that $\varphi^* \theta(X) = \theta(T)\varphi^*$. Hence

$$\int_0^t (\varphi_s^* \theta(X)\Phi) \, ds = \int_0^t (j_s^* \theta(T)\Phi) \, ds$$

$$= I_0^t \theta(X) \varphi^* \Phi$$

$$= (j_t^* \varphi^* \Phi) - (j_0^* \varphi^* \Phi)$$

$$= \varphi_t^* \Phi - \Phi.$$

Using the fundamental theorem of calculus, we obtain the following relations for a smooth family σ:

$$\int_a^b \dot{\sigma}_t \; dt = \sigma_b - \sigma_a \,, \qquad a, b \in \mathbb{R}$$

and

$$\left(\int_a \sigma\right)^{\cdot} (t, x) = \sigma(t, x), \qquad t \in \mathbb{R}, \; x \in M.$$

From the above relations, we get

$$\varphi_s^* \theta(X) \Phi = (\varphi_t^* \Phi - \Phi)_s^{\cdot} = (\varphi_t^* \Phi)_s^{\cdot}.$$

Thus, $\theta(X)\Phi = \varphi_0^* \, \theta(X)\Phi = (\varphi_t^* \Phi)_0^{\cdot}$ which completes the proof.

Oriented *n*-manifold

For an oriented n-manifold Ma graded δ-stable ideal $A_M(\mathbb{R} \times M) \subset A(\mathbb{R} \times M)$is defined as follows:

If $carr\ \Phi \cap (K \times M)$ is compact for all closed, finite intervals K, then $\Phi \in A_M(\mathbb{R} \times M)$.

Assume that $\mathbb{R}$ is oriented by the one-form $\delta t$and $\mathbb{R} \times M$ is the product orientation. Consider I as a finite open interval $(a, b) \subset \mathbb{R}$and let $j_a, j_b : M \to \mathbb{R} \times M$ be the inclusions opposite a and b. Then, $\overline{carr\ \Omega \cap (I \times M)}$ is compact for $\Omega \in A_M^{n+1}(\mathbb{R} \times M)$ (Cheng and Yau, 1975). Consequently, we can form the integral $\int_{I \times M} \Omega$.

Lemma 3. If $\Omega \in A_M^{n+1}(\mathbb{R} \times M)$, then $\int_{I \times M} \Omega = \int_M I_a^b \; i(T)\Omega$.

Proof. Let Lbe a compact subset of $\mathbb{R}^n$ and let M be an oriented n-manifold. By using a finite partition of unity in M, wereduce it to the case $M = \mathbb{R}^n$and $carr\ \Omega \subset \mathbb{R} \times L$. Assume that $\{e_1, \cdots, e_n\}$ is a positive basis of $\mathbb{R}^n$.Then $\delta t \wedge e^{*1} \wedge \cdots \wedge e^{*n}$ isa positive $(n+1)$-form in $\mathbb{R} \times \mathbb{R}^n$. If$f \in \mathcal{S}(\mathbb{R}^{n+1})$ and $carr\ f \subset \mathbb{R} \times L$, then we obtain

$$\Omega = f \cdot \delta t \wedge e^{*1} \wedge \cdots \wedge e^{*n}.$$

Also, we get $i(T)\Omega = f \cdot e^{*1} \wedge \cdots \wedge e^{*n}$ and it follows that

$$\left(I_a^b \; i(T)\Omega\right)(x) = \left(\int_a^b f(t, x) dt\right) e^{*1} \wedge \cdots \wedge e^{*n}.$$

Thus,
$$\int_{\mathbb{R}^n} I_a^b \; i(T)\Omega = \int_{\mathbb{R}^n} \int_a^b f(t, x) \; dt \; dx^1 \cdots dx^n$$
$$= \int_{I \times \mathbb{R}^n} \Omega.$$

Hence, $\int_{I \times M} \Omega = \int_M I_a^b \; i(T)\Omega$ for $\Omega \in A_M^{n+1}(\mathbb{R} \times M)$ which completes the proof.

Theorem 2. If M is an oriented n-manifold and $\Phi \in A_M^n(\mathbb{R} \times M)$, then

$$\int_{I\times M} \delta\Phi = \int_M j_b^*\Phi - \int_M j_a^*\Phi.$$

Proof. Let M be an oriented n-manifold and $\Phi \in A_M^n(\mathbb{R} \times M)$. Assume that the vector field T on $\mathbb{R} \times M$given by

$$T(s,x) = \left(\frac{d}{dt}, 0\right), \; s \in \mathbb{R}, x \in M.$$

The operator $I_a^b \circ i(T)$ is determined by T and given by

$$I_a^b \circ i(T): A^p(\mathbb{R} \times M) \to A^{p-1}(M).$$

The above operator evidently restricts to the following operator

$$I_a^b \circ i(T): A_M^p(\mathbb{R} \times M) \to A_C^{p-1}(M).$$

We know that if $\Omega \in A_M^{n+1}(\mathbb{R} \times M)$, then $\int_{I\times M}\Omega = \int_M I_a^b \; i(T)\Omega$. Consequently, we obtain

$$\int_{I\times\mathbb{R}^n} \delta\Phi = \int_M \left(I_a^b \; i(T)\right)\delta\Phi.$$

It follows that

$$I_a^b \; i(T)\delta\Phi = j_b^*\Phi - j_a^*\Phi - \delta I_a^b \; i(T)\Phi.$$

Since $\Phi \in A_M^n(\mathbb{R} \times M)$ and $I_a^b i(T)\Phi \in A_C^{n-1}(M)$, so we have

$$\int_M \delta I_a^b \; i(T)\Phi = 0.$$

Thus, $\int_{I\times M}\delta\Phi = \int_M j_b^*\Phi - \int_M j_a^*\Phi$ which completes the proof.

Theorem 3. Let U be a neighbourhood of the closed unit-ball $\bar{B}$ and $\Phi \in A^n(U)$. Then,

$$\int_B \delta\Phi = \int_S i^*\Phi \qquad \text{(i)}$$

Proof. Assume that E is a vector space and$p$be a smooth function in E such that

$$p(x) = 1, |x| \leq 1 \text{ and } carr\; p \subset U.$$

If we replace Φ by $p \cdot \Phi$, then either side of (i) will not be changed.Since $p \cdot \Phi \in A_c^n(E)$, so we may assume that $\Phi \in A_c^n(E)$.

Again, consider q as a smooth function in Esuch that

$$q(x) = 1, |x| \leq \tfrac{1}{4}; \; q(x) = 0, |x| \geq \tfrac{1}{2}.$$

It follows that $i^*(1-q)\Phi = i^*\Phi$. Furthermore, since $q \cdot \Phi \in A_c^n(B)$, it is obvious that

$$\int_B \delta\Phi = \int_B \delta[(1-q)\cdot\Phi] + \int_B \delta[q\cdot\Phi] = \int_B \delta[(1-q)\cdot\Phi].$$

Therefore, if we replace Φ by $(1-q)\cdot\Phi$, neither side of (i) will be changed; that is, it will be sufficient to consider the case

$$\Phi(x) = 0, |x| \le \frac{1}{4}.$$

Thus, we have $\int_B \delta\Phi = \int_A \delta\Phi, A = \left\{x : \frac{1}{4} < |x| < 1\right\}$. Assume that the diffeomorphism $\alpha: \mathbb{R}^+ \times S \to E - \{0\}$ given by

$$\alpha(t,x) = tx \ \ (t \in \mathbb{R}^+, x \in S).$$

Since α preserves orientations, setting $I = \left(\frac{1}{4}, 1\right)$, we obtain

$$\int_B \delta\Phi = \int_A \delta\Phi = \int_{I\times S} \alpha^*\delta\Phi = \int_{I\times S} \delta(\alpha^*\Phi).$$

If M is an oriented n-manifold and $\Phi \in A_M^n(\mathbb{R}\times M)$, then

$$\int_{I\times M} \delta\Phi = \int_M j_b^*\Phi - \int_M j_a^*\Phi.$$

Also, since $i = \alpha \circ j_1$ and $j_{1/4}^*\alpha^*\Phi = 0$, we get

$$\begin{aligned}\int_B \delta\Phi &= \int_S j_1^*\alpha^*\Phi - \int_S j_{1/4}^*\alpha^*\Phi \\ &= \int_S i^*\Phi.\end{aligned}$$

Thus, $\int_B \delta\Phi = \int_S i^*\Phi$ and the theorem is proved.

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